where N is a constant. From Eqs. (5.12) and (5.14) one can conclude without any difficulty that the Born expansion is uniformly convergent for

$$cN < 1. \tag{5.15}$$

The convergence verifies the validity of the Born expansion in Eq. (5.11).

From Eqs. (3.19), (5.8), and (5.11) the final form for the radial function $Y_{mn}(y)$ can be expressed as

$$\begin{aligned} Y_{mn}(y) = (1/y)\, e^{i\alpha_n} F_n(\bar\eta, y) &+ \int_c^\infty dy'\, \tilde G_n(y,y')[f_n^u(\bar\eta, y') \\ &+ \sum_{i=1}^\infty \int_c^\infty dy_1 \cdots \int_c^\infty dy_i\, K_{nm}(y', y_1) \\ &\times K_{nm}(y_1,y_2)\cdots K_{nm}(y_{i-1},y_i) f_n^u(\bar\eta, y_i)]. \end{aligned} \tag{5.16}$$

Since the series is uniformly convergent, Eq. (5.16) is a valid expression. As $c \to 0$, the integral part reduces to zero and the function $y^{-1}e^{i\alpha_n}F_n(\bar\eta, y)$ is the zeroth-order approximation for the radial function $Y_{mn}(y)$.

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On the Decomposition of Direct Products of Irreducible Representations

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A lemma concerning irreducible representations contained in the decomposition of a direct product of irreducible representations of simply reducible groups is generalized to arbitrary decomposable unitary and nonunitary groups.

I. INTRODUCTION

In the application of group theory in physics the problem very often arises of decomposing a direct product of two irreducible representation into a sum of irreducible parts. In the theory of solid state physics such a decomposition is required in defining selection rules in scattering processes in magnetic and nonmagnetic crystals.[1,2] A classical example of this is the addition of angular momentum in quantum mechanics. Wigner,[3] using a classification of irreducible representations given by Frobenius and Schur, proved a lemma concerning irreducible representations contained in the decomposition of a direct product of irreducible representations of simply reducible groups. The three-dimensional rotation group is a simply reducible group, and, for example, the fact that the addition of integer angular momenta does not contain half-integer momenta can be deduced directly from Wigner's lemma.

The purpose of this work is to generalize Wigner's lemma. We first review the Frobenius and Schur classification of irreducible representations and Wigner's lemma for simply reducible groups. This lemma is then generalized to arbitrary decomposable unitary and nonunitary groups.

II. SIMPLY REDUCIBLE GROUPS

Let Δ^k denote the kth irreducible representation, and u the elements of a unitary group G. Frobenius and Schur have shown that the irreducible representations of the group G can be classified into three cases[4]:

Case A: $\Delta^k(u)$ is equivalent to $\Delta^k(u)^*$ and potentially real, i.e., can be brought into real form.

Case B: $\Delta^k(u)$ is equivalent to $\Delta^k(u)^*$ and pseudoreal, i.e., can not be brought into real form.

Case C: $\Delta^k(u)$ is not equivalent to $\Delta^k(u)^*$.

For Cases A and B, $\Delta^k(u)$ is equivalent to $\Delta^k(u)^*$:

$$\Delta^k(u)^* = \beta_k^{-1}(u)\beta_k$$

and

$$\beta_k\beta_k^* = C_k E,$$

where $C_k = +1$ or -1 for Cases A and B, respectively.

A group is called simply reducible if[3]:

(1) Every element is equivalent to its reciprocal.

(2) The direct product of any two irreducible representations contains no representation more than once.

The first condition means that all irreducible representations of a simply reducible group are either Case A or Case B.

The following lemma for simply reducible groups has been proven by Wigner[3]:

Lemma 1: The direct product of two Case A or two Case B irreducible representations of a simply reducible group contains only Case A irreducible representations; the direct product of a Case A and Case B irreducible representation contains only Case B irreducible representations.

III. UNITARY GROUPS

For an arbitrary decomposable unitary group G we prove the following lemma:

Lemma 2: The direct product of two Case A or two Case B irreducible representations of an arbitrary decomposable unitary group G does not contain Case B irreducible representations; the direct product of a Case A and a Case B irreducible representation does not contain Case A irreducible representations.

The direct product, for example, of two Case A irreducible representations contains only Case A or Case C irreducible representations, each representation possibly more than once. For simply reducible groups Lemma 2 is identical to Lemma 1.

Proof of Lemma 2: We take the direct product $\Delta(u) = \Delta^i(u) \times \Delta^j(u)$, where Δ^i and Δ^j are either Case A or Case B irreducible representations, that is,

$$\Delta^i(u)^* = \beta_i^{-1}\Delta^i(u)\beta_i, \qquad \beta_i\beta_i^* = C_iE,$$

$$\Delta^j(u)^* = \beta_j^{-1}\Delta^j(u)\beta_j, \qquad \beta_j\beta_j^* = C_jE. \tag{1}$$

We show that if the decomposition of the direct product contains the irreducible representation Δ^k equivalent to Δ^{k*},

$$\Delta^k(u)^* = \beta_k^{-1}\Delta^k(u)\beta_k, \qquad \beta_k\beta_k^* = C_kE, \tag{2}$$

then $C_k = C_iC_j$.

The direct product $\Delta(u)$ is decomposed via a similarity transformation with a unitary matrix U:

$$\Delta_r(u) = U^{-1}\Delta(u)U.$$

We assume that Δ_r is in the following form

$$\Delta_r = \begin{pmatrix} \Delta^k & & & & & \\ & \Delta^k & & & & \\ & & \ddots & & & \\ & & & \Delta^k & & \\ & & & & \Delta^p & \\ & & & & & \ddots \end{pmatrix},$$

where Δ^k appears n times and is assumed to be equivalent to Δ^{k*}, i.e., is either a Case A or Case B irreducible representation.

Using (1), we have

$$\Delta_r(u)^* = U^{-1}\Delta(u)^*U^* = [U^{-1}(\beta_i \times \beta_j)U^*]^{-1}\Delta_r(U)[U^{-1}(\beta_i \times \beta_j)U^*].$$

Denoting $U^{-1}(\beta_i \times \beta_j)U^*$ by β, we write the preceeding relation as

$$\Delta_r(u)^* = \beta^{-1}\Delta_r(u)\beta, \tag{3}$$

where $\beta\beta^* = C_iC_jE.$

β is subdivided into blocks corresponding in dimension to the irreducible representations appearing in Δ_r:

$$\beta = \begin{pmatrix} \beta_{11} & \beta_{12} & \cdots \\ \beta_{21} & \beta_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}.$$

From (3) we have for $i,j = 1, 2, \ldots, n$

$$\Delta^k(u)^* = \beta_{ij}^{-1}\Delta^k(u)\beta_{ij}. \tag{4}$$

The β_{ij} for $i \le n$ and $j > n$, and $j \le n$ and $i > n$, are zero for they connect nonequivalent irreducible representations. β therefore is of the form

$$\beta = \begin{pmatrix} \beta_{11} & \cdots & \beta_{1n} & \\ \vdots & & \vdots & \bigcirc \\ \beta_{n1} & \cdots & \beta_{nn} & \\ & & & \beta_{n+1,n+1}\cdots \\ & \bigcirc & & \vdots \end{pmatrix}.$$

We consider now only the submatrix of β containing the matrices β_{ij}, $i,j = 1, 2, \ldots, n$, and denote this by $\bar{\beta}$. From the properties of β we have

$$\begin{pmatrix} \Delta^k(u)^* & & \\ & \ddots & \\ & & \Delta^k(u)^* \end{pmatrix} = \bar{\beta}^{-1} \begin{pmatrix} \Delta^k(u) & & \\ & \ddots & \\ & & \Delta^k(u) \end{pmatrix} \bar{\beta} \tag{5}$$

and

$$\bar{\beta}\bar{\beta}^* = C_iC_jE. \tag{6}$$

We will show that $\bar{\beta}$ can be transformed into the quasidiagonal form:

$$\begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \ddots & \\ & & & \alpha \end{pmatrix}.$$

From (2) and (5) we then have $\Delta^{k*} = \alpha^{-1}\Delta^k\alpha$ and $\alpha\alpha^* = C_kE$, and from (6) that $\alpha\alpha^* = C_iC_jE$, thus giving $C_k = C_iC_j$ proving Lemma 2.

The matrix $\bar{\beta}$ of relation (5) is not unique. (5) will remain unchanged under any similarity transformation with a unitary matrix of the form $A \times E$, where E is of the same dimension as the irreducible representation Δ^k and A is an arbitrary unitary matrix of dimension n, the number of times Δ^k appears in (5).[5]

The matrix $\bar{\beta}$ can be replaced by

$$(A \times E)\bar{\beta} \tag{7}$$

without changing the form of relation (5).

We seek a matrix A that will put (7) in the required quasidiagonal form. To do this, we look at the structure of the matrix $\bar{\beta}$. From (4) for $i = j = 1$ and for a general i and j

$$\Delta^k(u)^* = \beta_{11}^{-1}\Delta^k(u)\beta_{11},$$
$$\Delta^k(u)^* = \beta_{ij}^{-1}\Delta^k(u)\beta_{ij}$$

from which we have

$$\beta_{ij}\beta_{11}^{-1}\Delta^k(u) = \Delta^k(u)\beta_{ij}\beta_{11}^{-1}$$

giving, by Schur's lemma,[6] $\beta_{ij} = \lambda_{ij}\beta_{11}$, where λ_{ij} is a constant. $\bar{\beta}$ can be written now as

$$\bar{\beta} = \begin{pmatrix} \lambda_{11}\beta_{11} & & \lambda_{1n}\beta_{11} \\ \vdots & & \vdots \\ \lambda_{n1}\beta_{11} & \cdots & \lambda_{nn}\beta_{11} \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & & \lambda_{nn} \end{pmatrix} \times \beta_{11} \equiv \lambda \times \beta_{11}.$$

Since both $\bar{\beta}$ and β_{11} are unitary matrices, λ is also unitary. Finally, by choosing $A = \lambda^{-1}$, (7) takes on the required quasidiagonal form and the proof of Lemma 2 is complete.

IV. NONUNITARY GROUPS

A nonunitary group M contains elements half of which are unitary and half antiunitary. The unitary elements form an invariant subgroup G of index two, and we can write M as

$$M = G + Ga_0,$$

where a_0 is a fixed antiunitary element.

Corepresentations D^k of a nonunitary group M are constructed in one of three ways depending on the following classification of the irreducible representations Δ^k of the unitary subgroup G[7]:

Type I: $\Delta^k(u)$ is equivalent to $\Delta^k(a_0^{-1}ua_0)^*$, $\Delta^k(a_0^{-1}ua_0)^* = \beta_k^{-1}\Delta^k(u)\beta_k$ and $\beta_k\beta_k^* = \Delta^k(a_0^2)$.

Type II: $\Delta^k(u)$ is equivalent to $\Delta^k(a_0^{-1}ua_0)^*$, $\Delta^k(a_0^{-1}ua_0)^* = \beta_k^{-1}\Delta^k(u)\beta_k$ but $\beta_k\beta_k^* = -\Delta^k(a_0^2)$.

Type III: $\Delta^k(u)$ is not equivalent to $\Delta^k(a_0^{-1}ua_0)^*$.

The three types of corepresentations corresponding to the above classification of the irreducible representation of the unitary subgroup G are[7]

Type I: $D^k(u) = \Delta^k(u), \quad D^k(ua_0) = \Delta^k(u)\beta_k.$

Type II:

$$D^k(u) = \begin{pmatrix} \Delta^k(u) & \\ & \Delta^k(u) \end{pmatrix}, \quad D^k(ua_0) = \begin{pmatrix} & \Delta^k(u)\beta_k \\ -\Delta^k(u)\beta_k & \end{pmatrix}. \tag{8}$$

Type III:

$$D^k(u) = \begin{pmatrix} \Delta^k(u) & \\ & \Delta^k(a_0^{-1}ua_0)^* \end{pmatrix}, \quad D^k(ua_0) = \begin{pmatrix} & \Delta^k(ua_0^2) \\ \Delta^k(a_0^{-1}ua_0)^* & \end{pmatrix}.$$

TABLE I: The number of times the corepresentation D^k is contained in the direct product $D^i \times D^j$, denoted by C_{ij}^k, is given in terms of the d_{ij}^k, the number of times the irreducible representation Δ^k is contained in the direct product $\Delta^i \times \Delta^j$. Primed suffices, as in $d_{i'j}$, denote that the irreducible representation $\Delta^i(a_0^{-1}ua_0)^*$ replaces $\Delta^i(u)$ in the direct product.

D^i	D^j	D^k	C_{ij}^k
I	I	I	d_{ij}^k
I	I	II	$\frac{1}{2}d_{ij}^k$
I	I	III	d_{ij}^k
I	II	I	$2d_{ij}^k$
I	II	II	d_{ij}^k
I	II	III	$2d_{ij}^k$
I	III	I	$d_{ij}^k + d_{ij'}^k$
I	III	II	$\frac{1}{2}d_{ij}^k + \frac{1}{2}d_{ij'}^k$
I	III	III	$d_{ij}^k + d_{ij'}^k$
II	II	I	$4d_{ij}^k$
II	II	II	$2d_{ij}^k$
II	II	III	$4d_{ij}^k$
II	III	I	$2d_{ij}^k + 2d_{ij'}^k$
II	III	II	$d_{ij}^k + d_{ij'}^k$
II	III	III	$2d_{ij}^k + 2d_{ij'}^k$
III	III	I	$d_{ij}^k + d^k{}_{ij'} + d_{i'j}^k + d_{i'j'}^k$
III	III	II	$\frac{1}{2}d_{ij}^k + \frac{1}{2}d_{ij'}^k + \frac{1}{2}d_{i'j}^k + \frac{1}{2}d_{i'j'}^k$
III	III	III	$d_{ij}^k + d_{ij'}^k + d_{i'j}^k + d_{i'j'}^k$

The decomposition of direct products of two corepresentations of a nonunitary group M can be analyzed in terms of the decomposition of direct products of irreducible representations of the unitary subgroup G.

Let C_{ij}^k be the number of times the corepresentation D^k is contained in the direct product $D^i \times D^j$. C_{ij}^k is calculated from[8]

$$C_{ij}^k = \frac{\sum_u \chi(D^i(u))\chi(D^j(u))\chi(D^k(u))^*}{\sum_u \chi(D^k(u))\chi(D^k(u))^*}, \tag{9}$$

where $\chi(D^i(u))$ is the trace of $D^i(u)$. The number of times an irreducible representation Δ^k of the subgroup G of M is contained in the direct product $\Delta^i \times \Delta^j$ is denoted by d_{ij}^k and calculated from

$$d_{ij}^k = (l_k/n)\sum_u \chi(\Delta^i(u))\chi(\Delta^j(u))\chi(\Delta^k(u))^*, \tag{10}$$

where l_k is the dimension of Δ^k and n the order of the group G.

By using the explicit form of the corepresentations (8), the C_{ij}^k defined by (9) can be written in terms of the d_{ij}^k defined by (10). The explicit form of the relation depends on the type of the corepresentation D^i, D^j, and D^k. The relations between the C_{ij}^k and the d_{ij}^k, taken from Ref. 9, are listed in Table I.

We prove the following lemmas:

Lemma 3: The direct product of two Type I or two Type II irreducible representations of the subgroup G of an arbitrary decomposable nonunitary group $M = G + Ga_0$ does not contain Type II irreducible representations; the direct product of a Type I and a Type II irreducible representation does not contain Type I irreducible representations.

Lemma 4: The direct product of two Type I or two Type II corepresentations of an arbitrary decomposable nonunitary group M does not contain Type II corepresentations; the direct product of a Type I and a Type II corepresentation does not contain Type I corepresentations.

Proof of Lemma 3: We take the direct product $\Delta = \Delta^i \times \Delta^j$, where Δ^i and Δ^j are each either Type I or Type II irreducible representations of the subgroup G of a nonunitary group M:

$$\Delta^i(a_0^{-1}ua_0)^* = \beta_i^{-1}\Delta^i(u)\beta_i, \quad \beta_i\beta_i^* = C_i\Delta^i(a_0^2),$$
$$\Delta^j(a_0^{-1}ua_0)^* = \beta_j^{-1}\Delta^j(u)\beta_j, \quad \beta_j\beta_j^* = C_j\Delta^j(a_0^2).$$

If the decomposition of the direct product contains the irreducible representation $\Delta^k(u)$ equivalent to $\Delta^k(a_0^{-1}ua_0)^*$, possibly more than once,

$$\Delta^k(a_0^{-1}ua_0)^* = \beta_k^{-1}\Delta^k(u)\beta_k, \quad \beta_k\beta_k^* = C_k\Delta^k(a_0^2),$$

then $C_k = C_iC_j$. The remainder of this proof is parallel to the proof of Lemma 2.

The proof of Lemma 4 follows immediately from Lemma 3 and Table I.

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The Symmetric Group and the Gel'fand Basis of $U(3)$. Generalizations of the Dirac Identity

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It is shown that the symmetrization of N particle states by means of the orthogonal units of the algebra of the symmetric group S_N yields the Gel'fand basis states of the irreducible representations of $U(3)$. The existence of generalizations of the Dirac identity is demonstrated, and a connection between the symmetrized two- and three-body exchange operators and the invariants of $U(3)$ is established.

INTRODUCTION

The study of the unitary group $U(3)$ and, more generally, of $U(n)$ is of great interest to present day physics. Most well known is the successful classification of the elementary particles according to the octet model as proposed by Gell–Mann and Ne'eman[1] in 1962. A physically different application of the theory of the unitary groups has been to the many particle system. In fact, a great deal of the development of the theory—associated with the names of Racah and Wigner[2]—has been done towards the goal of classifying the electronic states in the atom. More recently, the theory of the unitary groups has been used to obtain approximate solutions of the nuclear many-body problem.[3]

The study of the many-body system leads, in a rather natural manner, to consideration of the operations which permute the particles and, thus, to the introduction of the symmetric group S_N. The connection between the two groups $U(n)$ and S_N has been known since the work of Young and Frobenius around 1900. Later, recognizing the importance of the concepts for quantum mechanics, Weyl[4] continued research along these lines and laid the foundation for our present understanding of the subject. He formulated the concept of duality and gave it an expression in a number of theorems. These early investigations have been concerned with the irreducible representations and have used the characters as tools. It was only within the past decade that a systematic investigation of the basis states has been taken up, pursued mainly by Biedenharn[5,6] and also by Moshinsky[7,8] and their collaborators. Yet, the relevance of the symmetric group for the Gel'fand[9] states has been considered to a limited extent only. Moshinsky[10] showed that a certain class of Gel'fand states had a definite permutational symmetry, and Ciftan and Biedenharn[11] and Ciftan[12] used the concept of "hooks" (which originally has its proper meaning in the symmetric group) to construct the Gel'fand states of $U(4)$.

In the present paper we show that the duality between $U(n)$ and S_N can be extended to the individual basis states defined by the subgroup decomposition[6] $U(n) \supset U(n-1) \supset \cdots \supset U(1)$ on the one side and by an analogous chain on the other side. It will be shown that the Gel'fand states can be obtained by use of operations of S_N only, thus supplying a link to the understanding of the hook structure concept for the unitary groups. In addition to their transformation properties under the unitary groups, the Gel'fand states will be seen to transform like the basis states of the irreducible representations of S_N. The situation will be pictured by introducing a "combined Young–Weyl tableau."

As a first step we shall demonstrate the existence of generalizations of the Dirac identity[13] which emerge naturally by considering the operations of both groups, $U(n)$ and S_N, in the same space. In this way we are led to explicit expressions for the fully symmetrized Majorana operator and the analogous three-body exchange operator in terms of the invariants of $U(3)$.