Algebraic construction of one-dimensional quasiperiodic tilings

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(Received 28 June 1988; accepted for publication 19 October 1988)

For an arbitrary one-dimensional quasiperiodic tiling constructed via the *grid method* with periodically spaced grids, an algebraic equation is derived for the positions of the vertices dependent on a single variable, the cardinal position of the vertices.

I. INTRODUCTION

Interest in quasiperiodic tiling predates the 1984 discovery by Shechtman *et al.*[1] of an alloy of aluminum and manganese that exhibits an electron diffraction pattern with icosahedral symmetry. In 1981 de Bruijn[2] developed a *grid method* to construct the prototypic two-dimensional quasiperiodic Penrose tilings,[3–5] and showed that these tilings could also be constructed by a *projection method* from a higher-dimensional space. Mackay[6,7] constructed a three-dimensional icosahedral quasiperiodic tiling using a pair of *unit cells*. The projection method was generalized to the construction of three-dimensional icosahedral quasiperiodic tilings by Kramer and Neri.[8] With the discovery by Shechtman *et al.*[1] much work has been published on the construction of three-dimensional quasiperiodic tilings using grid methods[9–11] and projection methods.[12–20] Gahler and Rhyner[11] have shown the equivalence of the two methods. We shall consider in this paper one-dimensional quasiperiodic tilings constructed via the grid method with periodically spaced grids. Those tilings constructed using quasiperiodically spaced grids or tilings generated by inflation rules not obtainable by the projection method[21] will not be considered.

In 1986, using special sequences of ones and zeros introduced by de Bruijn,[22] Litvin and Litvin[23] derived an algebraic equation for the positions of the vertices of a one-dimensional quasiperiodic tiling consisting of two basic tilings with the equation dependent on a single index, the cardinal position of the vertices. In this paper, we shall derive an analogous algebraic equation for the positions of the vertices of a one-dimensional quasiperiodic tiling consisting of an arbitrary number of basic tilings. In Sec. II we define an arbitrary one-dimensional quasiperiodic tiling using the grid method and state the algebraic equation for the vertices of this tiling. In Sec. III we give an inductive proof of this equation. In Sec. IV we compare this work with the results of Ref. 23.

II. ARBITRARY ONE-DIMENSIONAL QUASIPERIODIC TILING

We construct an arbitrary one-dimensional quasiperiodic tiling with p basic tilings using the grid method as follows: On a number line we plot the p set of points

$$\{nt_i + \gamma_i \mid n \in Z\}, \quad i = 1,2,\ldots,p, \tag{1}$$

where we shall assume that the ratios t_i/t_j, $i \neq j$, $i,j = 1,2,\ldots,p$, are irrational and the γ_i are constants such that $\gamma_i < t_i$ for $i = 1,2,\ldots,p$. This construct divides the number line into segments that can be characterized by a p-tuple of integers $(M_1,M_2,\ldots,M_p)$, where the $(M_1,M_2,\ldots,M_p)^{\text{th}}$ segment is defined by

$$\{x \mid M_1t_1 \leqslant x < (M_1+1)t_1 \cap M_2t_2 \leqslant x < (M_2+1)t_2 \cap \cdots \cap M_pt_p \leqslant x < (M_p+1)t_p\}. \tag{2}$$

For each p-tuple $(M_1,M_2,\ldots,M_p)$ determined above we construct on a second number line for a given p-tuple of real numbers $(a_1,a_2,\ldots,a_p)$ the point

$$X(M_1,M_2,\ldots,M_p) = M_1a_1 + M_2a_2 + \cdots + M_pa_p. \tag{3}$$

The problem that we consider is the derivation of an algebraic equation dependent on a single index, the cardinal position of the vertices, for the set of points defined by Eq. (3) for all p-tuples of integers $(M_1,M_2,\ldots,M_p)$ defined by Eqs. (1) and (2). We shall prove the following theorem.

Theorem: Let $N = M_1 + M_2 + \cdots + M_p$ be the cardinal position of the point $X(M_1,M_2,\ldots,M_p)$ defined by Eq. (3). Then the position $X(N)$ is given by

$$X(N) = \sum_{j=1}^{p} \left\lfloor \left(N + 1 + \sum_{m=1}^{p} \frac{\gamma_m - \gamma_j}{t_m}\right)\left(\sum_{k=1}^{p} \frac{t_j}{t_k}\right)^{-1} \right\rfloor a_j, \tag{4}$$

where $\lfloor\ \rfloor$ denotes the integer function, $\lfloor y \rfloor$ is the greatest integer less than or equal to y.

III. PROOF OF EQ. (4)

We shall give an inductive proof of Eq. (4). We shall prove that the theorem is correct for the cases of $p = 2$ and $p = 3$, and then for an arbitrary integer g, prove that it is correct for the case $p = g + 1$ assuming the equation is valid for the case $p = g$.

A. $p=2$

The vertices of the tilings in the case $p = 2$ are given by

$$X(M_1,M_2) = M_1a_1 + M_2a_2. \tag{3$'$}$$

We subdivide this set of points into two subsets; the first, the vertices at the end of tilings of length a_1 and the second, the vertices at the end of tilings of length a_2. We denote the points of these two subsets, respectively, as $X(N_1)$ and $X(N_2)$, where N_1 and N_2 denote the cardinal positions of the vertices in the tiling. We have that

$$X(N_1) = M_1a_1 + \lfloor (M_1t_1 + \gamma_1 - \gamma_2)/t_2 \rfloor a_2, \tag{5a}$$

$$X(N_2) = \lfloor (M_2t_2 + \gamma_2 - \gamma_1)/t_1 \rfloor a_1 + M_2a_2, \tag{5b}$$

where

 0022-2488/89/030740-04$02.50

$$N_1 = M_1 + \lfloor (M_1 t_1 + \gamma_1 - \gamma_2)/t_2 \rfloor, \quad (6a)$$

$$N_2 = M_2 + \lfloor (M_2 t_2 + \gamma_2 - \gamma_1)/t_1 \rfloor. \quad (6b)$$

We can invert Eqs. (6): Since $\lfloor y \rfloor = y - \Delta$, where $0 \leqslant \Delta < 1$, we can rewrite Eq. (6a) as

$$N_1 = M_1 + (M_1 t_1 + \gamma_1 - \gamma_2)/t_2 - \Delta$$

and consequently,

$$N_1 + 1 = M_1[(t_1 + t_2)/t_2] + (\gamma_1 - \gamma_2)/t_2 + (1 - \Delta),$$

$$(N_1 + 1)\frac{t_2}{t_1 + t_2} + \left(\frac{\gamma_2 - \gamma_1}{t_1 + t_2}\right) = M_1 + (1 - \Delta)\frac{t_2}{t_1 + t_2}.$$

Since M_1 is an integer and $0 \leqslant (1 - \Delta) t_2/(t_1 + t_2) < 1$, we have on taking the integer function of both sides of the previous equation that

$$M_1 = \lfloor (N_1 + 1)[t_2/(t_1 + t_2)] + (\gamma_2 - \gamma_1)/(t_1 + t_2) \rfloor,$$

$$M_1 = \left\lfloor \left(N_1 + 1 + \frac{\gamma_2 - \gamma_1}{t_2}\right)\left(\sum_{k=1}^{2} \frac{t_1}{t_k}\right)^{-1} \right\rfloor. \quad (7a)$$

In an analogous manner we derive from Eq. (6b) that

$$M_2 = \left\lfloor \left(N_2 + 1 + \frac{\gamma_1 - \gamma_2}{t_1}\right)\left(\sum_{k=1}^{2} \frac{t_2}{t_k}\right)^{-1} \right\rfloor. \quad (7b)$$

The coefficient of a_2 in Eq. (5a) is, using Eq. (6a), equal to $N_1 - M_1$, and using Eq. (7a) and the relationship that $-\lfloor y \rfloor = \lfloor -y \rfloor + 1$, we have

$$N_1 - M_1 = N_1 + 1 + \left\lfloor -\left(N_1 + 1 + \frac{\gamma_2 - \gamma_1}{t_2}\right) \times \left(\sum_{k=1}^{2} \frac{t_1}{t_k}\right)^{-1} \right\rfloor,$$

$$N_1 - M_1 = \left\lfloor \left(N_1 + 1 + \frac{\gamma_1 - \gamma_2}{t_1}\right)\left(\sum_{k=1}^{2} \frac{t_2}{t_k}\right)^{-1} \right\rfloor. \quad (8a)$$

The coefficient of a_1 in Eq. (5b) is, using Eq. (6b), equal to $N_2 - M_2$, and using Eq. (7b) we have

$$N_2 - M_2 = \left\lfloor \left(N_2 + 1 + \frac{\gamma_2 - \gamma_1}{t_2}\right)\left(\sum_{k=1}^{2} \frac{t_1}{t_k}\right)^{-1} \right\rfloor. \quad (8b)$$

We can now rewrite Eqs. (5) as

$$X(N_1) = \left\lfloor \left(N_1 + 1 + \frac{\gamma_2 - \gamma_1}{t_2}\right)\left(\sum_{k=1}^{2} \frac{t_1}{t_k}\right)^{-1} \right\rfloor a_1 + \left\lfloor \left(N_1 + 1 + \frac{\gamma_1 - \gamma_2}{t_1}\right)\left(\sum_{k=1}^{2} \frac{t_2}{t_k}\right)^{-1} \right\rfloor a_2,$$

$$X(N_2) = \left\lfloor \left(N_2 + 1 + \frac{\gamma_2 - \gamma_1}{t_2}\right)\left(\sum_{k=1}^{2} \frac{t_1}{t_k}\right)^{-1} \right\rfloor a_1 + \left\lfloor \left(N_2 + 1 + \frac{\gamma_1 - \gamma_2}{t_1}\right)\left(\sum_{k=1}^{2} \frac{t_2}{t_k}\right)^{-1} \right\rfloor a_2$$

and since the cardinal positions of the two subsets of vertices are mutually exclusive we can write a single equation for the positions of the vertices

$$X(N) = \left\lfloor \left(N + 1 + \frac{\gamma_2 - \gamma_1}{t_2}\right)\left(\sum_{k=1}^{2} \frac{t_1}{t_k}\right)^{-1} \right\rfloor a_1 + \left\lfloor \left(N + 1 + \frac{\gamma_1 - \gamma_2}{t_1}\right)\left(\sum_{k=1}^{2} \frac{t_2}{t_k}\right)^{-1} \right\rfloor a_2,$$

which proves Eq. (4) for the case where $p = 2$.

B. $p=3$

The vertices of the tilings in the case $p = 3$ are given by

$$X(M_1, M_2, M_3) = M_1 a_1 + M_2 a_2 + M_3 a_3. \quad (3'')$$

We subdivide this set of points into three subsets; the first, the vertices at the end of tilings of length a_1, the second, the vertices at the end of tilings of length a_2, and the third, the vertices at the end of tilings of length a_3. We denote these subsets of points, respectively, as $X(N_1)$, $X(N_2)$, and $X(N_3)$, where N_1, N_2, and N_3 denote the cardinal positions of the vertices in the tiling. We have that

$$X(N_1) = M_1 a_1 + \lfloor (M_1 t_1 + \gamma_1 - \gamma_2)/t_2 \rfloor a_2 + \lfloor (M_1 t_1 + \gamma_1 - \gamma_3)/t_3 \rfloor a_3, \quad (9a)$$

$$X(N_2) = \lfloor (M_2 t_2 + \gamma_2 - \gamma_1)/t_1 \rfloor a_1 + M_2 a_2 + \lfloor (M_2 t_2 + \gamma_2 - \gamma_3)/t_3 \rfloor a_3, \quad (9b)$$

$$X(N_3) = \lfloor (M_3 t_3 + \gamma_3 - \gamma_1)/t_1 \rfloor a_1 + \lfloor (M_3 t_3 + \gamma_3 - \gamma_2)/t_2 \rfloor a_2 + M_3 a_3, \quad (9c)$$

where

$$N_1 = M_1 + \lfloor (M_1 t_1 + \gamma_1 - \gamma_2)/t_2 \rfloor + \lfloor (M_1 t_1 + \gamma_1 - \gamma_3)/t_3 \rfloor, \quad (10a)$$

$$N_2 = M_2 + \lfloor (M_2 t_2 + \gamma_2 - \gamma_1)/t_1 \rfloor + \lfloor (M_2 t_2 + \gamma_2 - \gamma_3)/t_3 \rfloor, \quad (10b)$$

$$N_3 = M_3 + \lfloor (M_3 t_3 + \gamma_3 - \gamma_1)/t_1 \rfloor + \lfloor (M_3 t_3 + \gamma_3 - \gamma_2)/t_2 \rfloor. \quad (10c)$$

We can invert Eqs. (10): Since[24]

$$\lfloor \alpha + \beta \rfloor - 1 \leqslant \lfloor \alpha \rfloor + \lfloor \beta \rfloor \leqslant \lfloor \alpha + \beta \rfloor, \quad (11)$$

from Eq. (10a) we have

$$\left\lfloor M_1\left(\frac{t_1}{t_2} + \frac{t_1}{t_3}\right) + \frac{\gamma_1 - \gamma_2}{t_2} + \frac{\gamma_1 - \gamma_3}{t_3} \right\rfloor - 1 \leqslant N_1 - M_1 \leqslant \left\lfloor M_1\left(\frac{t_1}{t_2} + \frac{t_1}{t_3}\right) + \frac{\gamma_1 - \gamma_2}{t_2} + \frac{\gamma_1 - \gamma_3}{t_3} \right\rfloor.$$

From this it follows that

$$M_1 - 1 \leqslant \left\lfloor \left(N_1 + 1 + \frac{\gamma_2 - \gamma_1}{t_2} + \frac{\gamma_3 - \gamma_1}{t_3}\right) \times \left(\sum_{k=1}^{3} \frac{t_1}{t_k}\right)^{-1} \right\rfloor \leqslant M_1.$$

Since $M_1 - 1$ and M_1 are consecutive integers we have

$$M_1 = \left\lfloor \left(N_1 + 1 + \frac{\gamma_2 - \gamma_1}{t_2} + \frac{\gamma_3 - \gamma_1}{t_3}\right) \times \left(\sum_{k=1}^{3} \frac{t_1}{t_k}\right)^{-1} \right\rfloor + I, \quad (12)$$

where $I = 0$ or 1. This constant I is determined by taking the limit of Eq. (12) then t_3 goes to infinity. In this limit we obtain the $p = 2$ case and Eq. (12) must become identical with Eq. (7a). We find that $I = 0$ and

$$M_1 = \left\lfloor \left(N_1 + 1 + \frac{\gamma_2 - \gamma_1}{t_2} + \frac{\gamma_3 - \gamma_1}{t_3}\right)\left(\sum_{k=1}^{3} \frac{t_1}{t_k}\right)^{-1} \right\rfloor. \quad (13a)$$

In an analogous manner we invert Eq. (10b) and (10c) and obtain

$$M_2 = \left\lfloor \left(N_2 + 1 + \frac{\gamma_1 - \gamma_2}{t_1} + \frac{\gamma_3 - \gamma_2}{t_3}\right)\left(\sum_{k=1}^{3} \frac{t_2}{t_k}\right)^{-1} \right\rfloor, \tag{13b}$$

$$M_3 = \left\lfloor \left(N_3 + 1 + \frac{\gamma_1 - \gamma_3}{t_1} + \frac{\gamma_2 - \gamma_3}{t_2}\right)\left(\sum_{k=1}^{3} \frac{t_3}{t_k}\right)^{-1} \right\rfloor. \tag{13c}$$

Equations (13) are then substituted in Eq. (9). Denoting the coefficient of a_2 in Eq. (9a) by C_{12}, we have

$$C_{12} = \left\lfloor \left\lfloor \left(N_1 + 1 + \frac{\gamma_2 - \gamma_1}{t_2} + \frac{\gamma_3 - \gamma_1}{t_3}\right) \times \left(\sum_{k=1}^{3} \frac{t_1}{t_k}\right)^{-1} \right\rfloor \frac{t_1}{t_2} + \frac{\gamma_1 - \gamma_2}{t_2} \right\rfloor,$$

$$C_{12} = \left\lfloor \left(N_1 + 1 + \frac{\gamma_1 - \gamma_2}{t_1} + \frac{\gamma_3 - \gamma_2}{t_3}\right) \times \left(\sum_{k=1}^{3} \frac{t_2}{t_k}\right)^{-1} - \Delta \frac{t_1}{t_2} \right\rfloor,$$

$$C_{12} = \left\lfloor \left(N_1 + 1 + \frac{\gamma_1 - \gamma_2}{t_1} + \frac{\gamma_3 - \gamma_2}{t_3}\right) \times \left(\sum_{k=1}^{3} \frac{t_2}{t_k}\right)^{-1} \right\rfloor + I,$$

where I is some integer. On taking the limit of t_3 going to infinity we obtain the $p = 2$ case and this coefficient must then become identical with Eq. (8a). Consequently, $I = 0$ and

$$C_{12} = \left\lfloor \left(N_1 + 1 + \frac{\gamma_1 - \gamma_2}{t_1} + \frac{\gamma_3 - \gamma_2}{t_3}\right)\left(\sum_{k=1}^{3} \frac{t_2}{t_k}\right)^{-1} \right\rfloor.$$

In an analogous manner the coefficients C_{ij} of a_j in Eqs. (9), $i = 1,2,3$ corresponding to Eqs. (9a), (9b), and (9c), respectively, and $j = 1,2,3$, can be shown to be given by

$$C_{ij} = \left\lfloor \left(N_i + 1 + \sum_{m=1}^{3} \frac{\gamma_m - \gamma_j}{t_m}\right)\left(\sum_{k=1}^{3} \frac{t_j}{t_k}\right)^{-1} \right\rfloor.$$

Since the cardinal positions N_1, N_2, and N_3 of Eqs. (9) are mutually exclusive, we then can write a single equation for the positions of the vertices given by Eqs. (9),

$$X(N) = \sum_{j=1}^{3} \left\lfloor \left(N + 1 + \sum_{m=1}^{3} \frac{\gamma_m - \gamma_j}{t_m}\right)\left(\sum_{k=1}^{3} \frac{t_j}{t_k}\right)^{-1} \right\rfloor a_j, \tag{14}$$

proving the theorem, Eq. (4), for the case $p = 3$.

C. $p = g + 1$

In the general case we subdivide the set of points representing the vertices of the tiling into p subsets $X(N_i)$, $i = 1,2,...,p$, where N_i is the cardinal positions of the vertices at the end of tilings of length a_i. We have

$$X(N_i) = \sum_{j=1}^{p} C_{ij} a_j, \quad i = 1,2,...,p, \tag{15}$$

where

$$C_{ij} = \lfloor (M_i t_i + \gamma_i - \gamma_j) t_j \rfloor, \quad i,j = 1,2,...,p, \tag{16}$$

and

$$N_i = \sum_{j=1}^{p} C_{ij}, \tag{17}$$

Assuming that the theorem, Eq. (4), is valid for the case $p = g$, we have for the case $p = g$ that

$$C_{ij} = \left\lfloor \left(N_i + 1 + \sum_{m=1}^{g} \frac{\gamma_m - \gamma_j}{t_m}\right)\left(\sum_{k=1}^{g} \frac{t_j}{t_k}\right)^{-1} \right\rfloor, \quad i,j = 1,2,...,g, \tag{18}$$

and in particular that $M_i = C_{ii}$ for $i = 1,2,...,g$.

For the case $p = g + 1$, from Eqs. (16) and (17) we have

$$N_i = \sum_{j=1}^{g+1} \left\lfloor \frac{M_i t_i + \gamma_i - \gamma_j}{t_j} \right\rfloor,$$

and therefore

$$N_i + 1 = M_i \sum_{j=1}^{g+1} \frac{t_i}{t_j} + \sum_{j=1}^{g+1} \frac{\gamma_i - \gamma_j}{t_j} + 1 - \sum_{j=1}^{g+1} \Delta_j,$$

where Δ_j is defined by

$$\lfloor (M_i t_i + \gamma_i - \gamma_j)/t_j \rfloor = (M_i t_i + \gamma_i - \gamma_j)/t_j - \Delta_j.$$

It follows that

$$\left\lfloor \left(N_i + 1 + \sum_{j=1}^{g+1} \frac{\gamma_j - \gamma_i}{t_j}\right)\left(\sum_{k=1}^{g+1} \frac{t_i}{t_k}\right)^{-1} \right\rfloor = M_i + I, \tag{19}$$

where I is some integer. Equation (19) in the limit that t_q goes to infinity, $q \neq i$, must become the expression for $M_i = C_{ii}$ given by Eq. (18), for $i = j$, in the case $p = g$. Consequently, $I = 0$ and for the case $p = g + 1$,

$$M_i = \left\lfloor \left(N_i + 1 + \sum_{m=1}^{g+1} \frac{\gamma_m - \gamma_i}{t_m}\right)\left(\sum_{k=1}^{g+1} \frac{t_i}{t_k}\right)^{-1} \right\rfloor.$$

Substituting this into Eq. (16) in the case of $p = g + 1$ we have

$$C_{ij} = \left\lfloor \left\lfloor \left(N_i + 1 + \sum_{m=1}^{g+1} \frac{\gamma_m - \gamma_i}{t_m}\right) \times \left(\sum_{k=1}^{g+1} \frac{t_i}{t_k}\right)^{-1} \right\rfloor \frac{t_i}{t_j} + \frac{\gamma_i - \gamma_j}{t_j} \right\rfloor,$$

$$C_{ij} = \left\lfloor \left\lfloor \left(N_i + 1 + \sum_{m=1}^{g+1} \frac{\gamma_m - \gamma_i}{t_m}\right) \times \left(\sum_{k=1}^{g+1} \frac{t_i}{t_k}\right)^{-1} \right\rfloor \frac{t_i}{t_j} + \frac{\gamma_i - \gamma_j}{t_j} - \Delta \frac{t_i}{t_j} \right\rfloor,$$

$$C_{ij} = \left\lfloor \left(N_i + 1 + \sum_{m=1}^{g+1} \frac{\gamma_m - \gamma_j}{t_m}\right)\left(\sum_{k=1}^{g+1} \frac{t_j}{t_k}\right)^{-1} - \Delta \frac{t_i}{t_j} \right\rfloor,$$

$$C_{ij} = \left\lfloor \left(N_i + 1 + \sum_{m=1}^{g+1} \frac{\gamma_m - \gamma_j}{t_m}\right)\left(\sum_{k=1}^{g+1} \frac{t_j}{t_k}\right)^{-1} \right\rfloor + I,$$

where I is some integer. This equation is the limit that t_q, $q \neq i,j$ goes to infinity must become the expression for C_{ij} given in Eq. (18) in the case $p = g$. Consequently, $I = 0$. Since the sets of cardinal positions N_i, $i = 1,2,...,g + 1$, are mutually exclusive, Eqs. (15), for $p = g + 1$ can be written as a single equation

$$X(N) = \sum_{j=1}^{g+1} \left\lfloor \left(N + 1 + \sum_{m=1}^{g+1} \frac{\gamma_m - \gamma_j}{t_m}\right)\left(\sum_{k=1}^{g+1} \frac{t_j}{t_k}\right)^{-1} \right\rfloor a_j,$$

proving the theorem given in Eq. (4).

IV. COMPARISON WITH THE RESULTS OF LITVIN AND LITVIN[23]

The positions $X'(N)$ of the vertices of a quasiperiodic tiling with $p=2$ basic tilings of lengths $a_1 = \sin\theta$ and $a_2 = \cos\theta$, constructed via a projection method was given in Eq. (9) of Ref. 23,

$$X'(N) = Na_2 + (\lfloor \gamma^* + N/\alpha^* \rfloor - \lfloor \gamma^* \rfloor)(a_1 - a_2),$$

where α^* and γ^* are constants. This can be rewritten as

$$X'(N) = \lfloor N/\alpha^* + \gamma^* - \lfloor \gamma^* \rfloor \rfloor a_1 + (N - \lfloor N/\alpha^* + \gamma^* - \lfloor \gamma^* \rfloor \rfloor) a_2. \tag{20}$$

From Eq. (4) for $p=2$ we have

$$X(N) = \left\lfloor \left(N + 1 + \frac{\gamma_2 - \gamma_1}{t_2}\right)\left(1 + \frac{t_1}{t_2}\right)^{-1} \right\rfloor a_1 + \left\lfloor \left(N + 1 + \frac{\gamma_1 - \gamma_2}{t_1}\right)\left(1 + \frac{t_2}{t_1}\right)^{-1} \right\rfloor a_2,$$

from which we derive

$$X(N) = \left\lfloor \left(N + 1 + \frac{\gamma_2 - \gamma_1}{t_2}\right)\left(\frac{t_1 + t_2}{t_2}\right)^{-1} \right\rfloor a_1 + \left\{N - \left\lfloor \left(N + 1 + \frac{\gamma_2 - \gamma_1}{t_2}\right)\left(\frac{t_1 + t_2}{t_2}\right)^{-1} \right\rfloor \right\} a_2,$$

$$X(N) = \left\lfloor N\left(\frac{t_1 + t_2}{t_2}\right)^{-1} + \frac{t_2 + \gamma_2 - \gamma_1}{t_1 + t_2} \right\rfloor a_1 + \left\{N - \left\lfloor N\left(\frac{t_1 + t_2}{t_2}\right)^{-1} + \frac{t_2 + \gamma_2 - \gamma_1}{t_1 + t_2} \right\rfloor \right\} a_2,$$

and

$$X(N) = \left\lfloor N\left(\frac{t_1 + t_2}{t_2}\right)^{-1} + \frac{t_2 + \gamma_2 - \gamma_1}{t_1 + t_2} - \left\lfloor \frac{t_2 + \gamma_2 - \gamma_1}{t_1 + t_2} \right\rfloor \right\rfloor a_1 + \left\{N - \left\lfloor N\left(\frac{t_1 + t_2}{t_2}\right)^{-1} + \frac{t_2 + \gamma_2 - \gamma_1}{t_1 + t_2} - \left\lfloor \frac{t_2 + \gamma_2 - \gamma_1}{t_1 + t_2} \right\rfloor \right\rfloor \right\} a_2 + \left\lfloor \frac{t_2 + \gamma_2 - \gamma_1}{t_1 + t_2} \right\rfloor (a_1 + a_2). \tag{21}$$

Comparing Eqs. (20) and (21) we have

$$\alpha^* = (t_1 + t_2)/t_2,$$

$$\gamma^* = (t_2 + \gamma_2 - \gamma_1)/(t_1 + t_2),$$

and the positions of the vertices given by Eqs. (20) and (21) are related by a change in origin

$$X(N) = X'(N) + \lfloor (t_2 + \gamma_2 - \gamma_1)/(t_1 + t_2) \rfloor (a_1 + a_2).$$

ACKNOWLEDGMENT

The author would like to thank N. G. de Bruijn for correspondence.

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